Rank structured matrix operations

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Abstract. Classically, several numerical linear algebra problems are solved by means of tridiagonal (symmetric) and Hessenberg (unsymmetric) matrices. In this talk, it will be shown that a similar role can be played by semiseparable matrices. A matrix is called semiseparable if all submatrices that can be taken out of the lower triangular part (the main diagonal included) have maximum rank one. We will study how several matrix operations can be performed on such semiseparable, as well as more general rank structured matrices in an efficient and accurate way. We will illustrate the algorithms by means of several numerical examples.

1 Eigenvalue problems

When all eigenvalues (and eigenvectors) have to be computed for a given symmetric matrix of not too large size, the classical QR-algorithm is used, i.e., the symmetric matrix is transformed into a symmetric tridiagonal one by orthogonal similarity transformations and then the QR-algorithm is applied to this tridiagonal matrix. This transformation is performed because each step of the QR-algorithm applied to an $n \times n$ tridiagonal matrix only requires $O(n)$ flops. Moreover, the eigenvalues of the leading principal submatrices of the tridiagonal matrix are the Lanczos-Ritz values.

One could ask the question if there are other classes of matrices having interesting structures that have similar properties as the tridiagonal matrices. In [9], the necessary and sufficient conditions are derived such that the $k \times k$ leading principal submatrix of the matrix during the orthogonal similarity transformation process contains the Lanczos-Ritz values as eigenvalues just as in the tridiagonal case. These conditions lead naturally to other classes of structured matrices (semiseparable and diagonal-plus-semiseparable matrices). Moreover it turns out that each step of the QR-algorithm also needs only $O(n)$ operations.

In this talk, we will give an overview of the use of (symmetric) semiseparable matrices as an alternative to tridiagonal ones. There is a close relationship between the set of tridiagonal and semiseparable matrices. Indeed, the inverse of a nonsingular tridiagonal matrix is semiseparable and vice-versa.

Using orthogonal similarity transformations any symmetric matrix can be reduced into a symmetric semiseparable one [8]. The computational complexity of this reduction is of the same order as for the tridiagonal reduction. The implicit QR-algorithm on a semiseparable matrix can then be applied by a chasing technique [12]. Similar algorithms have been developed for the unsymmetric eigenproblem [11] and the singular value problem [10]. Also divide-and-conquer techniques [7], Lanczos-type algorithms [6], ... were designed using semiseparable matrices. Currently we are investigating how similar techniques can be applied to solve the generalized eigenvalue problem.

2 Using diagonal-plus-semiseparable matrices

When we allow more freedom in the reduction to rank structured form, one of the immediate choices is the introduction of a diagonal. The choice of the diagonal elements will influence the convergence behaviour when executing the reduction into diagonal-plus-semiseparable form [14].

3 More general rank structured matrices

We call a matrix *rank structured* if the ranks of certain submatrices starting from the lower-left matrix corner, as well as the ranks of certain submatrices starting from the upper-right matrix corner, are small compared to the matrix size. The class of rank structured matrices contains as special cases the classes of semiseparable matrices, unitary Hessenberg matrices, quasiseparable matrices and so on.

3.1 Representation

In order to specify algorithms for the class of rank structured matrices, we will first need an efficient representation. To this end we will use the Givens-weight representation: this is a generalization of the Givens-vector representation for semiseparable matrices [13], which is generalized to the case of an *arbitrary* rank structure.

3.2 Hessenberg reduction

Using the Givens-weight representation, we can devise an efficient algorithm to transform a rank structured matrix into a Hessenberg matrix by the use of unitary similarity transformations. The algorithm is important since the Hessenberg reduction process is commonly used as a first step to determine the eigenvalue spectrum of a matrix. We also show how the algorithm can be modified to transform the given matrix to bidiagonal form, by means of possibly different unitary row and column operations, i.e. by a reduction of the form $A \mapsto UAV$. This reduction can be used as a first step to compute the singular value decomposition of a matrix.

3.3 QR-factorization and QR-algorithm

When investigating matrix structures preserved under the QR-algorithm, under matrix inversion, under Schur complementation, the class of rank structured matrices turns out to maintain its structure under these operations [3, 4, 1, 2]. Taking the structure of the Q and R factor of the QR-factorization of a rank structured matrix into account, an efficient algorithm can be designed to compute this QR-factorization and solve the corresponding system of linear equations. These properties can also be used to devise a QR-algorithm where each step of the algorithm needs $O(n)$ floating point operations on a rank structured matrix of size n. Also a rank structured representation for the inverse can be computed in an efficient way.

4 Related topics

4.1 Zeros of polynomials

The zeros of a polynomial are classically computed as eigenvalues of the corresponding companion, comrade, ... matrix. Such a matrix belongs to the class of symmetric/unitary rank structured matrices plus a low rank matrix. This structure is maintained under the QR-algorithm. Hence, the implicit QR-algorithm for rank structured plus low rank matrices can be used to solve this problem efficiently and accurately.

4.2 Orthogonal rational functions

As the recurrence relation for orthogonal polynomials on the real line is related to symmetric tridiagonal matrices, the recurrence relation for orthogonal rational functions is connected to diagonal-plus-semiseparable matrices. The parameters of these relations can be computed by solving an inverse eigenvalue problem [5].

Notes and Comments. The aim of this extended abstract is to give an overview of the work of the MaSe-team (Matrices having Structure) that is done and still being done in the field of rank structured matrices. Hence, all the references are publications of members of the MaSe-team. This domain of research is very active over the last couple of years. One can also have a look at the work of Y. Eidelman and I. Gohberg; D. Bini and L. Gemignani; D. Fasino; S. Chandrasekaran, M. Gu, P. Dewilde and A.-J. van der Veen; R. Bevilacqua and G. Del Corso; E. Tyrtyshnikov; R. Nabben; L. Elsner; M. Fiedler; G. Meurant, ... There are also strong connections with hierarchical matrices, especially $\mathcal{H}^2$-matrices, developed by W. Hackbusch et al.

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