# Robust Boundary Conditions for Stochastic Incompletely Parabolic Systems of Equations

Markus Wahlsten & Jan Nordström

Division of Computational Mathematics,
Department of Mathematics,
Linköping University, SE-581 83 Linköping, Sweden

August 26, 2015

# Introduction

Consider the incompletely parabolic system of equations

$$\begin{array}{rcll} u_t + Au_x - \epsilon B u_{xx} & = & F(x,t,\xi) & 0 \le x \le 1, \quad t \ge 0 \\ H_0 u & = & g_0(t,\xi) & x = 0, \qquad t \ge 0 \\ H_1 u & = & g_1(t,\xi) & x = 1, \qquad t \ge 0 \\ u(x,0,\xi) & = & f(x,\xi) & 0 \le x, \qquad t = 0. \end{array} \tag{1}$$

The solution is represented by the vector $u = u(x,t,\xi)$ where, $\xi$ is a random variable. $A$ and $B$ are symmetric matrices. $H_0$ and $H_1$ are the boundary operators. $F$, $f$, $g_0$ and $g_1$ are the data to the problem.

# Outline

Outline

- Derivation of well-posed boundary conditions
- The study of the stochastic properties
- The study of a model problem
- Summary

# Derivation of well-posed boundary conditions

By ignoring the forcing function $F$, we multiply (1) by $u^T$ and integrat in space to obtain,

$$\begin{aligned} \|u\|_t^2 + 2\epsilon \int_0^1 u_x^T B u_x \, dx &= \left[u^T A u - 2\epsilon u^T B u_x\right]_{x=0} \\ &- \left[u^T A u - 2\epsilon u^T B u_x\right]_{x=1}, \end{aligned} \tag{2}$$

where $\|u\|^2 = \int_\Omega u^T u \, dx$. We now need to bound (2) by imposing boundary conditions.

# Derivation of well-posed boundary conditions

Lets consider only the left boundary terms ($LBT$), which can be diagonalized as

$$LBT = \left[u^T Au - 2\epsilon u^T Bu_x\right]_{x=0} = W_0^T \Lambda_D W_0, \qquad (3)$$

since $A$ and $B$ are symmetric. Let $W_0 = (W_0^+, W_0^-)$, hence (3) can be written as

$$LBT = (W_0^+)^T \Lambda_D^+ (W_0^+) + (W_0^-)^T \Lambda_D^- (W_0^-). \qquad (4)$$

Next, we impose the following general boundary condition in (4)

$$W_0^+ - R_0 W_0^- = 0, \qquad (5)$$

# Derivation of well-posed boundary conditions

The imposition of (5) in (4) gives

$$LBT = (W_0^-)^T (R_0^T \Lambda_D^+ R_0 + \Lambda_D^-)(W_0^-). \tag{6}$$

From (6) we conclude that

$$R_0^T \Lambda_D^+ R_0 + \Lambda_D^- \leq 0. \tag{7}$$

Finally, we end up with the general boundary operator

$$H_0 = \begin{bmatrix} X_+^T - \epsilon B^+ X^T \frac{\partial}{\partial x} \\ \epsilon Z_-^T X^T \frac{\partial}{\partial x} \end{bmatrix} - R_0 \begin{bmatrix} X_-^T - \epsilon B^- X^T \frac{\partial}{\partial x} \\ \epsilon Z_+^T X^T \frac{\partial}{\partial x} \end{bmatrix}. \tag{8}$$

where $R_0$ is chosen such that (7) holds.

# The study of the stochastic properties

We now focus on the stochastic properties of (1), formulated as,

$$
\begin{aligned}
u_t + Au_x - \epsilon Bu_{xx} &= F(x,t,\xi) &= \mathbb{E}[F](x,t) &+ \delta F(x,t,\xi) \\
H_0 u(0,t,\xi) &= g_0(t,\xi) &= \mathbb{E}[g_0](t) &+ \delta g_0(t,\xi) \\
H_1 u(1,t,\xi) &= g_1(t,\xi) &= \mathbb{E}[g_1](t) &+ \delta g_1(t,\xi) \\
u(x,0,\xi) &= f(x,\xi) &= \mathbb{E}[f](x) &+ \delta f(x,\xi).
\end{aligned}
\tag{9}
$$

Taking the expected value of (9) and defining $v = \mathbb{E}[u]$ we obtain,

$$
\begin{aligned}
v_t + Av_x - \epsilon Bv_{xx} &= \mathbb{E}[F](x,t) \\
H_0 v(0,t) &= \mathbb{E}[g_0](t) \\
H_1 v(1,t) &= \mathbb{E}[g_1](t) \\
v(x,0) &= \mathbb{E}[f](x).
\end{aligned}
\tag{10}
$$

# The study of the stochastic properties

Next, the difference between (9) and (10) together with the definition $e = u - v$ gives,

$$\begin{array}{rcl} e_t + Ae_x - \epsilon Be_{xx} & = & \delta F(x,t,\xi) \\ H_0 e(0,t,\xi) & = & \delta g_0(t,\xi) \\ H_1 e(1,t,\xi) & = & \delta g_1(t,\xi) \\ e(x,0,\xi) & = & \delta f(x,\xi). \end{array} \qquad (11)$$

The energy method applied to (11) gives (ignoring the right boundary)

$$\|e\|_t^2 + 2\epsilon \int_0^1 e_x^T Be_x \, dx = \begin{bmatrix} E_0^- \\ \delta g_0 \end{bmatrix}^T \begin{bmatrix} R_0^T \Lambda_D^+ R_0 + \Lambda_D^- & R_0^T \Lambda_D^+ \\ (R_0^T \Lambda_D^+)^T & \Lambda_D^+ \end{bmatrix} \begin{bmatrix} E_0^- \\ \delta g_0 \end{bmatrix} \qquad (12)$$

## The study of the stochastic properties

By taking the expected value of (12) and using the fact that

$$\mathbb{E}[\|e\|^2] = \|Var[u]\|_1 , \tag{13}$$

we find

$$\begin{aligned}\|Var[u]\|_t &+ 2\mathbb{E}[\epsilon \int_0^1 e_x^T B e_x \, dx] = \mathbb{E}[(E_0^-)^T \Lambda_D^- (E_0^-)] \\ &+ \mathbb{E}[(\delta g_0^-)^T \Lambda_D^+ (\delta g_0^-)] \\ &+ \mathbb{E}[(R_0 \delta g_0^+)^T \Lambda_D^+ (R_0 \delta g_0^+)] \\ &- 2\mathbb{E}[(R_0 \delta g_0^+)^T \Lambda_D^+ (\delta g_0^-)] \\ &+ \mathbb{E}[(\delta g_0^- - R_0 \delta g_0^+ + E_0^+)^T \Lambda_D^+ (R_0 E_0^-)].\end{aligned} \tag{14}$$

(14) implies that different types of boundary conditions (choices of $R_0$) gives different variance decay of the solution.

# The study of a model problem

Consider the simplest possible version of the general problem (1), where,

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

# The study of a model problem

For (15), the continuous boundary conditions are

$$\begin{aligned} W_0^+ - R_0 W_0^- &= g_0^+ - R_0 g_0^-, \\ W_1^- - R_1 W_1^+ &= g_1^- - R_1 g_1^+, \end{aligned} \qquad (16)$$

where,

$$\begin{aligned} W_0^+ &= \begin{bmatrix} +1 & 1 \end{bmatrix} u_{x=0} &-& \epsilon \begin{bmatrix} 0 & 1 \end{bmatrix} (u_x)_{x=0}, \\ W_0^- &= \begin{bmatrix} -1 & 1 \end{bmatrix} u_{x=0} &+& \epsilon \begin{bmatrix} 0 & 1 \end{bmatrix} (u_x)_{x=0}, \\ W_1^- &= \begin{bmatrix} -1 & 1 \end{bmatrix} u_{x=1} &+& \epsilon \begin{bmatrix} 0 & 1 \end{bmatrix} (u_x)_{x=1}, \\ W_1^+ &= \begin{bmatrix} +1 & 1 \end{bmatrix} u_{x=1} &-& \epsilon \begin{bmatrix} 0 & 1 \end{bmatrix} (u_x)_{x=1}. \end{aligned}$$

# Zero variance on the boundary

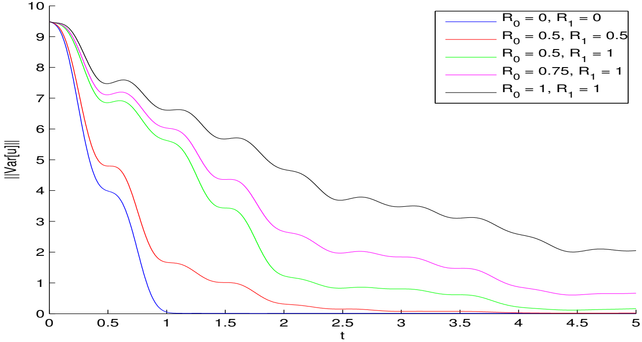


Figure : The $L_1$-norm of the variance as a function of time for a normally distributed $\xi$ for characteristic and non-characteristic boundary conditions when having perfect boundary knowledge.

# Decaying variance

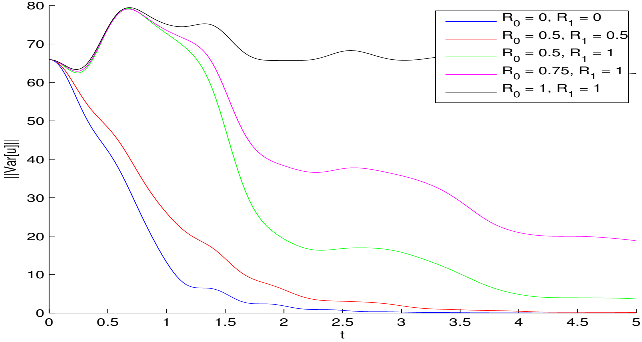


Figure : The $L_1$-norm of the variance as a function of time for a normally distributed $\xi$ for characteristic and non-characteristic boundary conditions when having decaying boundary data.

# Large non-decaying variance

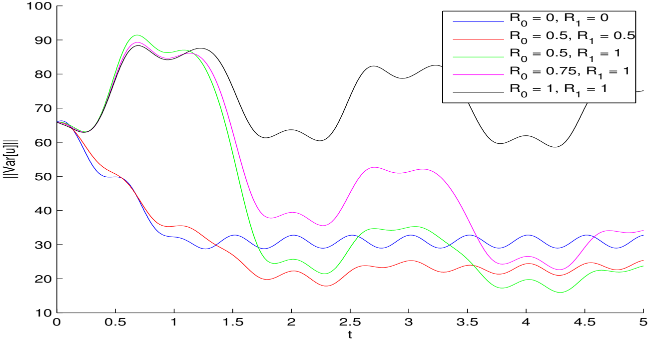


Figure : The $L_1$-norm of the variance as a function of time for a normally distributed $\xi$ for characteristic and non-characteristic boundary conditions when having large non-decaying boundary data.

# Summary

Summary

- Well-posed boundary conditions for an incompletely parabolic system of equations has been derived.
- The problem has been discretized using a finite difference scheme based on the SBP-SAT technique.
- An expression showing how the variance depends on the boundary conditions imposed has been derived.
- Numerical results show that generalized characteristic boundary conditions are generally a good choice in terms of variance minimization.