Parallel Algorithms for Standard and Generalized Triangular Sylvester-type Matrix Equations

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Abstract. We consider parallel algorithms for solving eight common standard and generalized triangular Sylvester-type matrix equation. Our parallel algorithms are based on explicit blocking, 2D block-cyclic data distribution of the matrices and wavefront-like traversal of the right hand side matrices while solving small matrix equations and updating the rest of the right hand side with respect to the subsolutions in level-3 operations. We apply the triangular solvers in condition estimation, developing parallel sep^{-1}-estimators. A resulting software package SCASY which contains general and triangular solvers and condition estimators is briefly presented.

Keywords: Sylvester-type matrix equations, Bartels–Stewart method, explicit blocking, level 3 BLAS, ScaLAPACK, condition estimation.

1 Introduction

We consider the following standard Sylvester-type matrix equations: the *continuous-time Sylvester equation* (SYCT)

$$AX - XB = C, \tag{1}$$

the discrete-time Sylvester equation (SYDT)

$$AXB^T - X = C, \tag{2}$$

the continuous-time Lyapunov equation (LYCT)

$$AX + XA^T = C, \tag{3}$$

and *the discrete-time Lyapunov equation* (LYDT)

$$AXA^T - X = C, \tag{4}$$

where A of size $m \times m$, B of size $n \times n$ and C of size $m \times n$ or $m \times m$ are general matrices with real entries. For LYCT/LYDT a symmetric right hand side C implies a symmetric solution X.

We also consider the following generalized Sylvester-type matrix equations: *the generalized coupled Sylvester equation* (GCSY)

$$(AX - YB,\ DX - YE) = (C,\ F), \tag{5}$$

where A and D of size $m \times m$, B and E of size $n \times n$ and C and F of size $m \times n$ are general matrices with real entries, *the generalized Sylvester equation* (GSYL)

$$AXB^T - CXD^T = E, \tag{6}$$

where A and C of size $m \times m$, B and D of size $n \times n$ and E of size $m \times n$ are general matrices with real entries, *the continuous-time generalized Lyapunov equation* (GLYCT)

$$AXE^T + EXA^T = C, \tag{7}$$

where A, E and C of size $m \times m$ are general matrices with real entries, and *the discrete-time generalized Lyapunov equation* (GLYDT)

$$AXA^T - EXE^T = C, \tag{8}$$

where A, E and C of size $m \times m$ are general matrices with real entries. For GLYCT and GLYDT a symmetric right hand side C implies a symmetric solution X.

Solvability conditions for equations (1)-(8) can be formulated in terms of the eigenvalues of the involved matrices (matrix pairs).

SYCT, LYCT and GCSY are called *one-sided* because the solution is multiplied by another matrix from one side only, and SYDT, LYDT, GSYL, GLYCT and GLYDT are, similarly, called *two-sided* [8, 9].

2 Parallel algorithms for triangular matrix equations

The parallel algorithms for SYCT presented in [11, 4, 3] were based on Bartels–Stewart's method [1], which consists of four basic steps. In this contribution, we focus on the third step which consists of solving a triangular matrix equation, i.e., a reduced matrix equation where the coefficients are in upper triangular real Schur form. For solving a triangular matrix equation we utilize explicit blocking and 2D block cyclic distribution of the matrices over a rectangular process grid, following the ScaLAPACK conventions [2]. The solution is computed via a wavefront-like traversal of block diagonals of the right hand side matrix where several solutions of diagonal subsystems are computed, broadcasted along the corresponding block row and column, and used in level-3 updates of the rest of the right hand side.

The algorithms are adopted to LYCT by wavefront-like traversal of the anti-diagonals of a symmetric right hand side matrix and only solving for the lower triangular part of the solution.

For the two-sided standard matrix equations SYDT/LYDT the main difference from the SYCT/LYCT cases are the need for an extra buffer for storing *intermediate sums of matrix products*. We illustrate with the following blocked SYDT system:

$$\begin{cases} A_{11}X_{11}B_{11}^T - X_{11} = C_{11} - A_{11}X_{12}B_{12}^T - A_{12}(X_{21}B_{11}^T + X_{22}B_{12}^T) \\ A_{11}X_{12}B_{22}^T - X_{12} = C_{12} - A_{12}X_{22}B_{22}^T \\ A_{22}X_{21}B_{11}^T - X_{21} = C_{21} - A_{22}X_{22}B_{12}^T \\ A_{22}X_{22}B_{22}^T - X_{22} = C_{22}. \end{cases} \tag{9}$$

From (9) we observe that work is saved by computing $X_{21}B_{11}^T + X_{22}B_{12}^T$ before multiplying with A_{12} and by computing $X_{22}B_{12}^T$ only once. Consequently, for SYDT/LYDT we broadcast each subsolution X_{ij} in block row i and a sum of matrix products in the corresponding block column j.

The generalized matrix equations are solved as follows: for GCSY the SYCT methodology is used except for the fact that we are now working with two equations at the same time. The methods of SYDT and LYDT are generalized for GSYL and GLYCT/GLYDT, respectively, in a similar fashion.

3 Condition estimators for triangular matrix equations

We utilize a general method [5, 6, 10] for estimating $\|A^{-1}\|_1$ for a square matrix A using reverse communication of $A^{-1}x$ and $A^{-T}x$, where $\|x\|_2 = 1$. In particular, for the SYCT this approach is based on the observation that SYCT is equivalent to the linear system $Z_{\text{SYCT}}x = c$, where $Z_{\text{SYCT}} = I_n \otimes A - B^T \otimes I_m$. This makes is possible to compute a lower bound of the inverse of the *separation between the matrices A and B* [13]

$$\text{sep}(A, B) = \inf_{\|X\|_F=1} \|AX - XB\|_F = \sigma_{min}(Z_{\text{SYCT}}) = \|Z_{\text{SYCT}}^{-1}\|_2^{-1}. \tag{10}$$

The quantity (10) is used frequently in perturbation theory and error bounds (see e.g. [7]). The cost for computing its exact value is $O(m^3n^3)$ flops by computing the singular value decomposition (SVD) of Z_{SYCT}. On the other side, $\text{sep}(A, B)$ can be estimated by solving a few (say five) triangular SYCT equations to only the cost $O(m^2 + n^2)$ flops [10]. This estimation method can be applied to any matrix equation by considering the corresponding Kronecker product representation of the associated Sylvester-type operator.

4 SCASY - a ScaLAPACK-style software package for Sylvester-type matrix equations

We have implemented parallel Bartels–Stewart-based general and triangular solvers for solving 42 sign and transpose variants of equations (1)-(8). Together with condition estimators based on the parallel triangular solvers, these routines will be made available and distributed through the SCASY library [12].

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References

1. R. H. Bartels and G. W. Stewart. Algorithm 432: The Solution of the Matrix Equation $AX - BX = C$. *Communications of the ACM*, 8:820–826, 1972.
2. L. S. Blackford, J. Choi, A. Cleary, E. D'Azevedo, J. W. Demmel, I. Dhillon, J. J. Dongarra, S. Hammarling, G. Henry, A. Petitet, K. Stanley, D. Walker, and R. C. Whaley. *ScaLAPACK Users' Guide.* SIAM, Philadelphia, PA, 1997.
3. R. Granat and B. Kågström. Evaluating Parallel Algorithms for Solving Sylvester-Type Matrix Equations: Direct Transformation-Based versus Iterative Matrix-Sign-Function-Based Methods. In K. Madsen In J. Dongarra and J. Wasniewski, editors, *PARA 2004*, pages 719–729. Lecture Notes in Computer Science, Springer-Verlag Heidelberg-Berlin, 2005.
4. R. Granat, B. Kågström, and P. Poromaa. Parallel ScaLAPACK-style Algorithms for Solving Continuous-Time Sylvester Equations. In H. et al Kosch, editor, *Euro-Par 2003 Parallel Processing,*, volume 2790, pages 800–809. Lecture Notes in Computer Science, Springer, 2003.
5. W.W. Hager. Condition estimates. *SIAM J. Sci. Statist. Comput.*, (3):311–316, 1984.
6. N. J. Higham. Fortran codes for estimating the one-norm of a real or complex matrix, with applications to condition estimation. *ACM Trans. of Math. Software*, 14(4):381–396, 1988.
7. N. J. Higham. Perturbation theory and backward error for $AX - XB = C$. *BIT*, 33(1):124–136, 1993.
8. I. Jonsson and B. Kågström. Recursive blocked algorithms for solving triangular systems. I. One-sided and coupled Sylvester-type matrix equations. *ACM Trans. Math. Software*, 28(4):392–415, 2002.
9. I. Jonsson and B. Kågström. Recursive blocked algorithms for solving triangular systems. II. Two-sided and generalized Sylvester and Lyapunov matrix equations. *ACM Trans. Math. Software*, 28(4):416–435, 2002.
10. B. Kågström and P. Poromaa. Distributed and shared memory block algorithms for the triangular Sylvester equation with sep^{-1} estimators. *SIAM J. Matrix Anal. Appl.*, 13(1):90–101, 1992.
11. P. Poromaa. Parallel Algorithms for Triangular Sylvester Equations: Design, Scheduling and Scalability Issues. In Kågström et al., editor, *Applied Parallel Computing. Large Scale Scientific and Industrial Problems*, volume 1541, pages 438–446. Springer Verlag, Lecture Notes in Computer Science, 1998.
12. SCASY - ScaLAPACK-style solvers for Sylvester-type matrix equations. See http://www.cs.umu.se/research/parallel/scasy.
13. G. W. Stewart and J.-G. Sun. *Matrix Perturbation Theory.* Academic Press, New York, 1990.