A Family of Runge-Kutta Restarters for Multistep Methods

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- *Runge–Kutta Starters for Multistep methods*, C. W. Gear, 1980.
- *A Runge–Kutta starter for a multistep method for differential-algebraic systems with discontinuous effects*, R. von Schwerin and H. G. Bock, 1994.

Restarting a multistep method

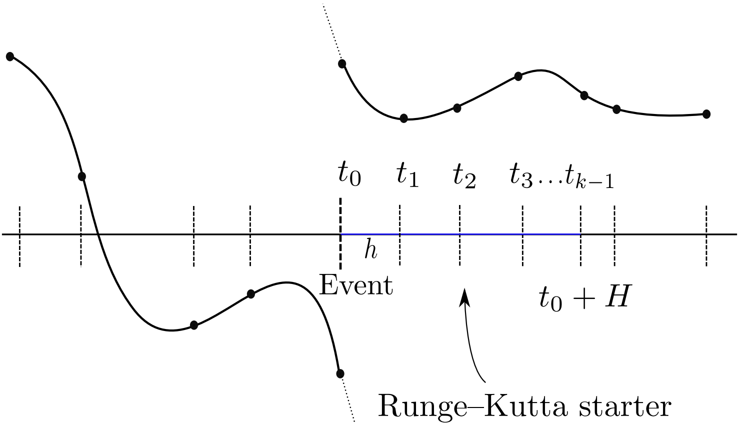
- Classical self-starting multistep method
- Apply several Runge–Kutta steps
- A single Runge–Kutta step

Goals of construction

For starting a k-step multistep method, the explicit Runge–Kutta starter is required to have

- Minimal number of internal stages.
- At least k-stage values of order k.
- Error estimation.
- Equidistant stages for starting the multistep method.

Runge–Kutta starter after a discontinuity



H: Runge–Kutta step size
h: Multistep step size

Explicit Runge–Kutta method

For initial value problem

$$y' = f(t, y), \qquad y(0) = y_0,$$

an s-stage explicit Runge–Kutta method is,

$$\begin{aligned}
g_i :=& y_0 + H \sum_{j=1}^{i} a_{ij} k_j, \qquad i = 1, \ldots, s, \\
k_i :=& f(t_0 + c_i H, g_i), \\
y_1 :=& y_0 + H \sum_{j=1}^{s} b_j k_j,
\end{aligned}$$

Internal order conditions

$k = 4$

$$\sum_j a_{ij} c_j = \frac{c_i^2}{2}, \qquad \sum_j a_{ij} c_j^2 = \frac{c_i^3}{3},$$

$$\sum_{jk} a_{ik} a_{kj} c_j = \frac{c_i^3}{6}, \qquad \sum_j a_{ij} c_j^3 = \frac{c_i^4}{4},$$

$$\sum_{jk} a_{ij} c_j a_{jk} c_k = \frac{c_i^4}{8}, \qquad \sum_{jk} a_{ij} a_{jk} c_k^2 = \frac{c_i^4}{12},$$

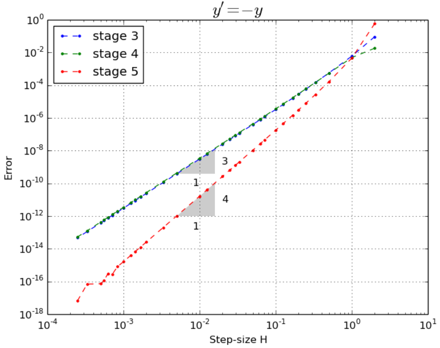
$$\sum_{jkl} a_{ij} a_{jk} a_{kl} c_l = \frac{c_i^4}{24},$$

For internal stage i.

An embedded third order Runge–Kutta starter

$$
\begin{array}{c|ccccc}
0 & & & & & \\
\frac{1}{2} & \frac{1}{6} & & & & \\
\frac{3}{4} & 0 & \frac{3}{4} & & & \\
1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & & \\
\frac{1}{2} & \frac{17}{72} & \frac{1}{6} & \frac{2}{9} & -\frac{1}{8} & \\
\hline
1 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

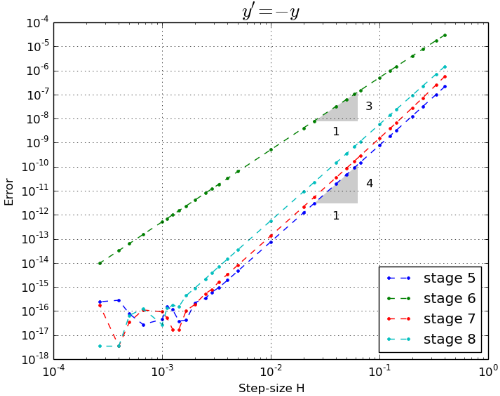
Order Plot



An embedded fourth order Runge–Kutta starter

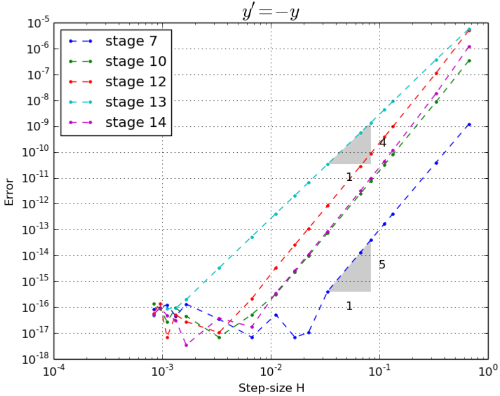
$$
\begin{array}{c|ccccccc}
0 & & & & & & & \\
\frac{1}{6} & \frac{1}{6} & & & & & & \\
\frac{1}{6} & 0 & \frac{1}{6} & & & & & \\
\frac{1}{3} & 0 & 0 & \frac{1}{3} & & & & \\
\frac{1}{3} & \frac{1}{18} & \frac{1}{9} & \frac{1}{9} & \frac{1}{18} & & & \\
1 & \frac{25}{10} & -3 & -3 & \frac{225}{100} & \frac{225}{100} & & \\
\frac{2}{3} & \frac{10}{45} & -\frac{8}{45} & -\frac{8}{45} & -\frac{4}{45} & \frac{13}{15} & \frac{1}{45} & \\
\hline
1 & -\frac{29}{1062} & \frac{83}{531} & \frac{83}{531} & \frac{83}{1062} & \frac{2}{531} & \frac{56}{531} & \frac{251}{531}
\end{array}
$$

Order Plot



$y' = -y$
Error
Step-size H
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{-15}
10^{-16}
10^{-17}
10^{-18}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
3
1
4
1
stage 5
stage 6
stage 7
stage 8

Order Plot



$y' = -y$
stage 7
stage 10
stage 12
stage 13
stage 14
Error
Step-size H
4
1
5
1

Implementation

Tests are conducted in Assimulo (Python):

(default simulation environment of JModelica.org)

Integrator: LSODAR (ODEPACK, FORTRAN)

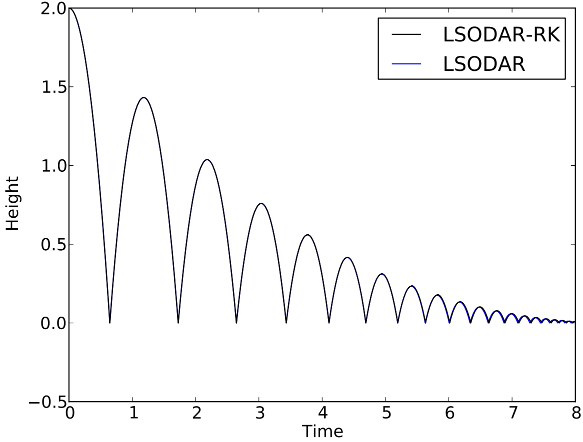
The bouncing ball test example

the example of a bouncing ball with linear damping $d = 0.1$,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -dy_2 + 9.81 \end{aligned}$$

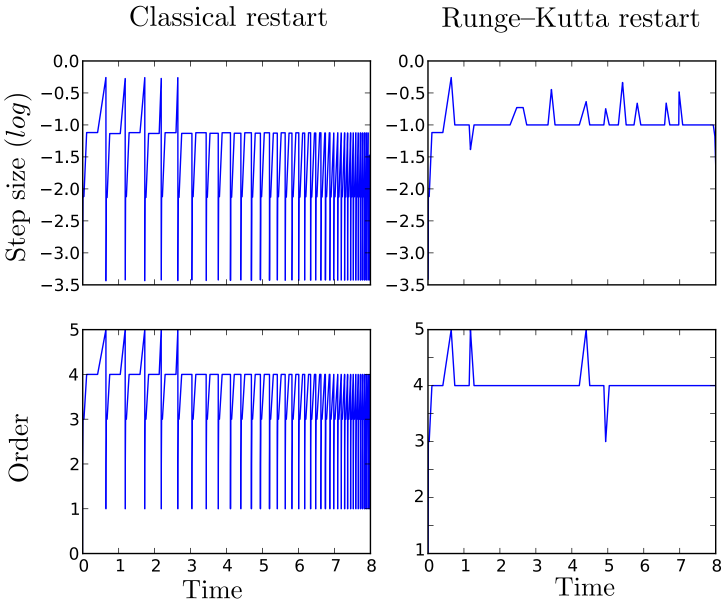
At the bouncing event the velocity $y_2(t^-)$ is altered to $y_2(t^+) = -cy_2(t^-)$.

Bouncing ball: simulation results



(damping: $d = 0.1$, coefficient of restitution $c = 0.88$)

Bouncing ball: Step size and order history



Conclusion and future work

- RKrestarter is a promising alternative in case of frequent events with small or no changes in the dynamics.
- Various restarting orders have been derived. Goal: No order drop after restarting.
- Step size control for the starter.

How to continue....

- Various strategies for choosing the initial step sizes have to be tested.
- Tests with complex physical models.

Thank you!